Zhu on elementary math-7

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The first sweetness of learning algebra — — — solving equations

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This article is suitable for students in fifth grade and above and their parents, primary school teachers, middle school teachers, young adults, and any mathematics educator who wants to learn elementary mathematics again.

After children understand algebraic formulas and algebraic operations (mainly addition, multiplication, and exponential operations, see the article "The First Obstacle in Mathematics Learning - Fractional Operations"), they can play with some speedy calculation tricks. What I like most is the simple games such as quick calculation 78×72=7×8×100+8× $2 = 5616$, 87×83=8×9×100+7×3=7221, etc, hoping that the children can get familiar with the operation like (10x+a)(10x+b).

In fact, when a+b=10,

$$(10x + a)(10x + b) = 100x^2 + 100x + ab = x(x + 1) \times 100 + ab.$$

From the above formula, it is easy to understand the principle of the above quick calculation. See if you can tell the answer of 65×65 now?

When I first learned algebraic expressions (I don’t remember whether I learned them myself or someone else taught me), what impressed me most was the joy brought by solving equations. Arithmetic word problems that can only be solved with great effort (who in China has never heard of the problem of a chicken and a rabbit in the same cage? This problem was listed in "Sun Tzu's Arithmetic Classic" more than 1,500 years ago), now can be solved relatively easier and directly by writing equations and then solving the equations. So much so that I am still asking: Why not learn how to solve equations first? I won’t challenge this problem here, but would rather share the joy of solving equations with you.

Don't worry about terms like linear equations, quadratic equations, system of quadratic equations, etc. We first state the properties of equality for equations and the principles of solving equations. Euclid (a name worth remembering by any scholar) wrote down more than two thousand years ago, and everyone used it very casually in the following years, the following axioms:

Properties of equality

$(1)((Invariant\ under\ addition) \qquad For\ any\ number\ C,$

$$A = B \iff A + C = B + C.$$

$(2)\ (Invariant\ under\ multiplication):\ For\ any\ C \neq 0,$

$$A = B \iff AC = BC.$$

$(3)\ (Symmetric\ property)\}:$

$$A = B \iff B = A.$$

And

$(4)(Transitive\ property) \qquad If\ A = B,\ and\ B = C,\ then$

$$A = C.$$

Example 1. We will see how to use above properties to solve equation: $11 = 2x + 3$.

Solution: *Using property (1), we have*

$$11 = 2x + 3 \iff 11 - 3 = 2x.$$

After simplifying, we have an equivalent equation: $8 = 2x$. *Using property (2) (by choosing* $C = \frac{1}{2}$*), we have* $4 = x$. *Using symmetric property (3), we finally derive* $x = 4$.

Smart students may argue that they immediately see $x = 4$, and do not care about the above detailed argument. We just want to point out: the above argument not only yields the solution, but also logically verifying that $x = 4$ is the ***only*** solution.

We argue: for human beings to solve equations, understanding the properties of equality sometime is even more important than understanding the classification of equations. Here is one example.

Example 2. Solve for x variable:

$$\frac{4}{x} - \frac{1}{3} = \frac{3}{x}.$$

Solution: *Once students are familiar with fractions, they can observe* $\frac{4}{x} = 4 \times \frac{1}{x}$, *and* $\frac{3}{x} = 3 \times \frac{1}{x}$.

Using (2.1-1), we have

$$\frac{4}{x} - \frac{1}{3} = \frac{3}{x} \quad \Leftrightarrow \quad \frac{1}{x} = \frac{1}{3}.$$

From the definition of the reciprocal number, we immediately know that $x = 3$.

Caution: In the above solution, we strongly discourage students to multiply both sides of the equation by $3x$ (in order to get rid of the denominators) . In the following, we will present examples to show that it is "dangerous" to multiply both sides of an equation by an unknow numbers.

After doing some practice questions, students may realize that equations like $ax + b = 0$ are easy to solve (for x value). We usually call such equations "first-degree equations" since the algebraic expression in the left is a first-degree polynomial of x. The other letters (a and b) are called parameters.

Let's look at an example of parametric equation that is more confusing.

Example 3: Find the value of x:

(1) $ax = 1$,

(2) $ax = 0$.

Solution: *(1). Case 1: If $a \neq 0$, then it has a reciprocal 1/a. Multiply both sides of the equation by 1/a, and we get $x = 1/a$.*

Case 2: If $a = 0$, then no matter what value x is, the equation (now becomes $0 \times x = 1$) does not hold ($0 \neq 1$), so the equation has no solution.

(2). Case 1: : If $a \neq 0$, then it has a reciprocal 1/a. Multiply both sides of the equation by 1/a, and we get $x = 0$.

Case 2: If $a = 0$, then no matter what value x is, the equation (now becomes $0 \times x = 0$) always holds ($0 = 0$), so any number is a solution to the equation.

The worst scenario is that students pay no attention to the value of the parameter a, and just solve the equation first. For example, by solving (1), they first get $x = 1/a$, and then analyze the result to conclude: when $a = 0$, the solution is meaningless! But wait: what does "the solution is meaningless" mean? When $a = 0$, it has no reciprocal at all. How can one get $x = 1/a$ at the first place? Similarly, solve (2) to get x=0/a, and then when a=0, you will be confused: is 0/0 actually the same as 0? --- It turns out these are really nonsense questions following from the wrong start. It's riddled with holes before asking questions, do you still need to think about it?

Finally, let's analyze how to solve the following equation with parameters.

Example 4: Find the value of x:

$$\frac{a}{x} - \frac{1}{2} = \frac{1}{2x}.$$

Solution：First we observe： $\frac{a}{x} = a \times \frac{1}{x}$, *and* $\frac{1}{2x} = \frac{1}{2} \times \frac{1}{x}$. *From property （1）, we obtain*

$$(a - \frac{1}{2})\frac{1}{x} = \frac{1}{2}.$$

Case 1，If $a \neq 1/2$, *then*

$$\frac{1}{x} = \frac{1}{2} \times \frac{1}{a - \frac{1}{2}} = \frac{1}{2a-1}.$$

Using the property of the reciprocal numbers, we have: $x = 2a - 1$.

Case 2，If $a = 1/2$, *for whatever value of* x, $0 \times \frac{1}{x} = 0 \neq \frac{1}{2}$. *That is: no solution to the equation!*

A mistake that is easy to overlook is to reduce the original equation to a "first-degree equation". Multiply "$2x$" on both sides of the equation and get: $2a - x = 1$. Then from the properties (1) and (3) we get the result: x=2a-1.

However, this result is wrong, because when a=1/2, you get x=0 is the solution to the equation. But 0 obviously cannot be the solution of the equation, because then one of the denominators of the fractions in the equation would be 0.

The reason for the error is that: if we multiply both sides of the equation by an uncertain term (that may be "0"), we may introduce an ``extraneous solution’’ to the original equation.

Solving equations can be a lot of fun and a lot of challenges. By following the rules, we will make fewer mistakes, experience less frustration, and enjoy algebra more.